XXV. *A new Method of finding Fluents by Continuation. By the Rev.* Samuel Vince, *A. M. F. R. S.*

Read July 6, 1786.

ART. I. Put $\dot{F} = \frac{x^m \dot{x}}{a^n + x^n} = x^{m-n}\dot{x} - a^n x^{m-2n}\dot{x} + a^{2n}x^{m-3n}\dot{x} -$ &c. $\pm \frac{a^{vn}x^{m-vn}\dot{x}}{a^n+x^n}$, then $F = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n+1}}{m-2n+1} + \frac{a^{2n}x^{m-3n+1}}{m-3n+1} -$ &c. $\pm W$, where W repreſents the fluent of the laſt term. Now $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} = \frac{x^m\dot{x} \times \overline{a^n+x^n}|^{1-r}}{a^n+x^n}$; hence $\int \frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} = \int \frac{x^m\dot{x} \times \overline{a^n+x^n}|^{1-r}}{a^n+x^n} = F \times \overline{a^n+x^n}|^{1-r} - \int F \times \frac{\overline{1-r}\,.\,nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^r}$ = (by ſubſtituting for F its value in the latter quantity) $F \times \overline{a^n+x^n}|^{1-r} - \frac{\overline{1-r}\,.\,n}{m-n+1} \times \int \frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} + \frac{\overline{1-r}\,.\,na^n}{m-2n+1} \times \int \frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^r} - \frac{\overline{1-r}\,.\,na^{2n}}{m-3n+1} \times \int \frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^r} +$ &c. $\pm \int W \times \frac{\overline{1-r}\,.\,nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^r}$; tranſpoſe $-\frac{\overline{1-r}\,.\,n}{m-n+1} \times \int \frac{x^m\dot{x}}{\overline{a^n+x^n}|^r}$ and divide both ſides of the equation by $\frac{m-rn+1}{m-n+1}$ and we have $\int \frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} = \frac{m-n+1}{m-rn+1} \times F \times \overline{a^n+x^n}|^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}\,.\,na^n}{m-2n+1} \times \int \frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^r} - \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}\,.\,na^{2n}}{m-3n+1} \times \int \frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^r} +$ &c. $\pm \frac{m-n+1}{m-rn+1} \times$

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$\int W \times \frac{\overline{1-r}.nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^r}$; now the fluent of the laſt term is $\mp\frac{m-n+1}{m-rn+1} \times W \times \overline{a^n+x^n}|^{1-r} \pm \frac{m-n+1}{m-rn+1} \times \int \frac{a^{vn}x^{m-vn}\dot{x}}{\overline{a^n+x^n}|^r}$; hence by ſubſtituting this quantity for the laſt term, it is manifeſt, that the firſt part $\mp \frac{m-n+1}{m-rn+1} \times W \times \overline{a^n+x^n}|^{1-r}$ will be deſtroyed by the laſt term of $\frac{m-n+1}{m-rn+1} \times F \times \overline{a^n+x^n}|^{1-r}$, when we ſubſtitute for F its value; hence if we put $M = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n-1}}{m-2n+1} + \frac{a^{2n}x^{m-3n+1}}{m-3n+1}$ – &c. omitting the laſt term $\pm$ W, we have $\int \frac{x^m\dot{x}}{\overline{x^n+x^n}|^r} = \frac{m-n+1}{m-rn+1} \times M \times \overline{a^n+x^n}|^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}.na^n}{m-2n+1} \times \int \frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^r} - \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}.na^{2n}}{m-3n+1} \times \int \frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^r} + \&c. \pm \frac{m-n+1}{m-rn+1} \times a^{vn} \times \int \frac{x^{m-vn}\dot{x}}{\overline{a^n+x^n}|^r}$; hence, if the fluent of the laſt term be given, we have the general law of continuation by which we may find the fluent of $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r}$. If the fluxion be $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r}$ all the terms after the firſt will be negative, and the laſt always poſitive.

Ex. 1. Given the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ to find the fluent of $\frac{x^{2s}\dot{x}}{\sqrt{1+x^2}}$.

Here $n=2$, $a=1$, $m=2s$, $r=\frac{1}{2}$, $M = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3}$ + &c. to $\pm x$,

$\pm x$, and the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ is the hyp. log. $x+\sqrt{1+x^2}$ which call Q; hence $\int\frac{x^{2s}\dot{x}}{\sqrt{1+x^2}}=\frac{2s-1}{2s}\times M\times\sqrt{1+x^2}+\frac{2s-1}{2s\,.\,2s-3}\times\int\frac{x^{2s-2}\dot{x}}{\sqrt{1+x^2}}-\frac{2s-1}{2s\,.\,2s-5}\times\int\frac{x^{2s-4}\dot{x}}{\sqrt{1+x^2}}+\text{\&c.}\pm\frac{2s-1}{2s}\times Q$.

If $s=1$, $\int\frac{x^2\dot{x}}{\sqrt{1+x^2}}=\frac{1}{2}\sqrt{1+x^2}\times x-\frac{1}{2}Q=\alpha$.

$s=2$, $\int\frac{x^4\dot{x}}{\sqrt{1+x^2}}=\frac{3}{4}\sqrt{1+x^2}\times\overline{\frac{x^3}{3}-x}+\frac{3}{4}\alpha+\frac{3}{4}Q=\beta$.

$s=3$, $\int\frac{x^6\dot{x}}{\sqrt{1+x^2}}=\frac{5}{6}\sqrt{1+x^2}\times\overline{\frac{x^5}{5}-\frac{x^3}{3}+x}+\frac{5}{6\,.\,3}\beta-\frac{5}{6}\alpha-\frac{5}{6}Q=\gamma$.

$s=4$, $\int\frac{x^8\dot{x}}{\sqrt{1+x^2}}=\frac{7}{8}\sqrt{1+x^2}\times\overline{\frac{x^7}{7}-\frac{x^5}{5}+\frac{x^3}{3}-x}+\frac{7}{8\,.\,5}\gamma-\frac{7}{8\,.\,3}\beta+\frac{7}{8}\alpha+\frac{7}{8}Q$.

&c. &c. &c.

Ex. 2. To find the fluent of $x^{\frac{s}{2}}\dot{x}\sqrt{2+x}$, given the fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$, and s an odd number.

Here $a=2$, $n=1$, $r=-\frac{1}{2}$, $\frac{s}{2}=m$, $m-vn=-\frac{1}{2}$, or $\frac{s}{2}-v=-\frac{1}{2}$; $\therefore v=\frac{s+1}{2}$, $M=\frac{2x^{\frac{s}{2}}}{s}-\frac{4x^{\frac{s}{2}-1}}{s-2}+\frac{8x^{\frac{s}{2}-2}}{s-4}-$ &c. and the fluent (Q) of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$ is $\pi+\sqrt{2x+x^2}$, where $\pi=$ hyp. log. $1+x+\sqrt{2x+x^2}$; hence $\int x^{\frac{s}{2}}\dot{x}\sqrt{2+x}=\frac{s}{s+3}\times\overline{2+x}|^{\frac{3}{2}}\times M+\frac{s}{s+3}\times\frac{6}{s-2}\times\int x^{\frac{s}{2}-1}\dot{x}\sqrt{2+x}-\frac{s}{s+3}\times\frac{12}{s-4}\times\int x^{\frac{s}{2}-2}\dot{x}\sqrt{2+x}+\frac{s}{s+3}\times\frac{24}{s-6}\times\int x^{\frac{s}{2}-3}\dot{x}\sqrt{2+x}$ &c. $\pm\frac{s}{s+3}\times 2^{\frac{s+1}{2}}\times\int x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$.

If

If $s=1$, $\int x^{\frac{1}{2}}\dot{x}\sqrt{2+x}=\frac{1}{4}\times\overline{2+x}|^{\frac{3}{2}}\times 2x^{\frac{1}{2}}-\frac{1}{2}Q=\alpha.$

$s=3$, $\int x^{\frac{3}{2}}\dot{x}\sqrt{2+x}=\frac{1}{2}\times\overline{2+x}|^{\frac{3}{2}}\times\overline{\frac{2x^{\frac{3}{2}}}{3}-4x^{\frac{1}{2}}}+3\alpha+2Q=\beta.$

$s=5$, $\int x^{\frac{5}{2}}\dot{x}\sqrt{2+x}=\frac{5}{8}\times\overline{2+x}|^{\frac{3}{2}}\times\overline{\frac{2x^{\frac{5}{2}}}{5}-\frac{4x^{\frac{3}{2}}}{3}+8x^{\frac{1}{2}}}+\frac{5}{4}\beta-$
$\frac{15}{2}\alpha-5Q=\gamma.$

$s=7$, $\int x^{\frac{7}{2}}\dot{x}\sqrt{2+x}=\frac{7}{10}\times\overline{2+x}|^{\frac{3}{2}}\times\overline{\frac{2x^{\frac{7}{2}}}{7}-\frac{4x^{\frac{5}{2}}}{5}+\frac{8x^{\frac{3}{2}}}{3}-16x^{\frac{1}{2}}}+$
$\frac{21}{25}\gamma-\frac{14}{5}\beta+\frac{84}{5}\alpha+\frac{56}{5}Q.$

&c. &c. &c.

II. Let $\dot{F}\frac{x^n\dot{x}}{a+bx^m+x^{2m}}=x^{n-2m}\dot{x}-Px^{n-3m}\dot{x}+Qx^{n-4m}\dot{x}-$ &c. $\pm$ $\frac{Vx^{n-sm}\dot{x}}{a+bx^m+x^{2m}}\pm\frac{Wx^{n-\overline{s+1}\cdot m}\dot{x}}{a+bx^m+x^{2m}}$, then $F=\frac{x^{n-2m+1}}{n-2m+1}-\frac{Px^{n-3m-1}}{n-3m+1}+\frac{Qx^{n-4m+1}}{n-4m+1}$ $-$ &c. $\pm T\pm U$, where T and U are put for the fluents of the two laft terms, and P, Q, &c. for the co-efficients arifing from the divifion. Now, $\int\frac{x^n\dot{x}}{\sqrt{a+bx^m+x^{2m}}}=\int\frac{x^n\dot{x}\times\overline{a+bx^m+x^{2m}}|^{1-r}}{a+bx^m+x^{2m}}=$

$F\times\overline{a+bx^m+x^{2m}}|^{1-r}-\int F+\frac{\overline{1-r}\cdot mbx^{m-1}\dot{x}+\overline{1-r}\cdot 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}=$ (by fubftituting for F its value in the latter quantity, and putting A, B, C, &c for the co-efficients which arife in confequence thereof) $F\times\overline{a+bx^m+x^{2m}}|^{1-r}-A\times\int\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}+B\times$

$\int\frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}-C\times\int\frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}+$ &c.

$$\mp\int T\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$$

$\mp\int U\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$; hence by transposition and division we have $\int\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}=\frac{1}{1+A}\times F\times\overline{a+bx^m+x^{2m}}|^{1-r}$ $+\frac{B}{1+A}\times\int\frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}-\frac{C}{1+A}\times\int\frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}+$ &c.

$$\mp\int\frac{T}{1+A}\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$$

$\mp\int\frac{U}{1+A}\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$. Now the fluents of these two last terms are $\mp\frac{T}{1+A}\times\overline{a+bx^m+x^{2m}}|^{1-r}\pm\frac{V}{1+A}$ $\int\frac{x^{n-3m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$ and $\mp\frac{U}{1+A}\times\overline{a+bx^m+x^{2m}}|^{1-r}\pm\frac{W}{1+A}\int\frac{x^{n-\overline{s+1}\,.\,m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$ respectively; hence, by substituting these for the last term, it is manifest that $\mp\frac{T}{1+A}\times\overline{a+bx^m+x^{2m}}|^{1-r}$ and $\mp\frac{U}{1+A}$ $\times\overline{a+bx^m+x^{2m}}|^{1-r}$ will be destroyed by the two last terms of $\frac{1}{1+A}\times F\times\overline{a+bx^m+x^{2m}}|^{1-r}$ when we substitute for F its value; hence, if we put $M=\frac{x^{n-2m-1}}{n-2m+1}-\frac{Px^{n-3m+1}}{n-3m+1}+\frac{Qx^{n-4m+1}}{n-4m+1}-$ &c. omitting the two last terms $\pm$ T and $\pm$ U, we shall have $\int\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}=\frac{1}{1+A}\times M\times\overline{a+bx^m+x^{2m}}|^{1-r}+\frac{B}{1+A}\times$ $\int\frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}-\frac{C}{1+A}\times\int\frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}+$ &c. $\pm\frac{V}{1+A}\times$

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$\int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} \pm \frac{W}{1+A} \times \frac{x^{\overline{n-s+1}.m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$; hence if the two last fluents be given, we have the general law of continuation up to $\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$ in the same manner as before.

III. In general, if we proceed as in the two last articles, we shall find $\int \frac{x^n\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} = \frac{M}{P} \times \overline{a+bx^m+\&c.\ x^{tm}}|^{1-r} + \frac{A}{P} \times \int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} + \frac{B}{P} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} + \&c. \pm \frac{T}{P} \times \int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} \pm \frac{V}{P} \times \int \frac{x^{\overline{n-s+1}.m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tn}}|^r} \pm \&c.$ where the number of these last terms is t, and $M = \frac{x^{n-tm+1}}{n-tm+1} - \frac{Qx^{\overline{n-t+1}.m-1}}{\overline{n-t+1}.m+1}$ + &c. omitting, as before, the terms at the end arising from the remainders. Hence if the last t fluents be given, we can by continuation find the required fluent.

Because the division of $\frac{x^n\dot{x}}{a+bx^m+\&c.\ x^{tm}}$ may be expressed by an *ascending* series $x^n\dot{x} - Qx^{n+m}\dot{x} + Rx^{n+2m}\dot{x} - \&c.$ it is manifest, that by the same method we may continue the fluents downwards as well as upwards.

IV. Let $\dot{F} = \frac{x^n\dot{x}}{1-x} = -x^{n-1}\dot{x} - x^{n-2}\dot{x} - x^{n-3}\dot{x} - \&c. - x^{n-r}\dot{x} + \frac{x^{n-r}\dot{x}}{1-x}$, then $F = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1} + W$, where W is the fluent of the last term. Now $\frac{x^n\dot{x}}{\sqrt{1-x^2}} = \frac{x^n\dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}}$,

 hence

hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = \int \frac{x^n \dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}} = \mathrm{F} \times \sqrt{\frac{1-x}{1+x}} + \int \mathrm{F} \times \frac{\dot{x}}{\sqrt{1+x^2} \times \overline{1+x}} = \mathrm{F} \times \sqrt{\frac{1-x}{1+x}} - \int \frac{x^n \dot{x}}{n\sqrt{1-x^2} \times \overline{1+x}} - \int \frac{x^{n-1} \dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2} \times \overline{1+x}} - \int \frac{x^{n-2} \dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2} \times \overline{1+x}} - \&c. - \int \frac{x^{n-r+1} \dot{x}}{\overline{n-r+1} \,.\, \sqrt{1-x^2} \times \overline{1+x}} + \int \mathrm{W} \times \frac{\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}}$. But

$$\begin{array}{lll}
\frac{x^n \dot{x}}{\overline{1-x^2} \times \overline{1+x}} = & \frac{x^{n-1} \dot{x}}{n\sqrt{1-x^2}} - \frac{x^{n-2} \dot{x}}{n\sqrt{1-x^2}} + \frac{x^{n-3} \dot{x}}{n\sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r} \dot{x}}{n\sqrt{1-x^2}} \pm \frac{x^{n-r} \dot{x}}{n\sqrt{1-x^2} \times \overline{1+x}} \\
\frac{x^{n-1} \dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = & \frac{x^{n-2} \dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2}} - \frac{x^{n-3} \dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2}} + \&c. \pm \frac{x^{n-r} \dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2}} \mp \frac{x^{n-r} \dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2} \times \overline{1+x}} \\
\frac{x^{n-2} \dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = & \frac{x^{n-3} \dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r} \dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2}} \pm \frac{x^{n-r} \dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2} \times \overline{1+x}} \\
\&c. & \&c. \qquad \&c. \\
\frac{x^{n-r+1} \dot{x}}{\overline{n-r+1} \,.\, \sqrt{1-x^2} \times \overline{1+x}} = & \frac{x^{n-r} \dot{x}}{\overline{n-r+1} \,.\, \sqrt{1+x^2}} - \frac{x^{n-r} \dot{x}}{\overline{n-r+1} \,.\, \sqrt{1-x^2} \times \overline{1+x}}.
\end{array}$$

Hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = \mathrm{F} \times \sqrt{\frac{1-x}{1+x}} - \frac{1}{n} \times \int \frac{x^{n-1} \dot{x}}{\sqrt{1-x^2}} + \overline{\frac{1}{n} - \frac{1}{n-1}} \times \int \frac{x^{n-2} \dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3} \dot{x}}{\sqrt{1-x}} + \&c. \pm \overline{\frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1}} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}} \mp \overline{\frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1}} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} + \int \mathrm{W} \times \frac{\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}}$. Now $\int \mathrm{W} \times \frac{\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = -\mathrm{W} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}} + \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}}$; also $\int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = -x^{n-r} \times \sqrt{\frac{1-x}{1+x}} + \overline{n-r} \times \int \frac{x^{n-r-1} \dot{x}}{\sqrt{1-x^2}} - \overline{n-r} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}}$; hence, by ſubſtituting theſe quantities in the two laſt terms, it is manifeſt that $-\mathrm{W} \times \overline{\frac{1-x}{1-x}}\Big|^{\frac{1}{2}}$ will be deſtroyed by the laſt term of

F ×

$F \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}$ when we ſubſtitute for F its value; therefore, if we put $M = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1}$, we ſhall have

$$\int \frac{x^n \dot{x}}{\sqrt{1+2}} = \overline{M \pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1}} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}} - \frac{1}{n} \times \int \frac{x^{n-1}\dot{x}}{\sqrt{1-x^2}}$$

$$+ \overline{\frac{1}{n} - \frac{1}{n-1}} \times \int \frac{x^{n-2}\dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3}\dot{x}}{\sqrt{1-x^2}} + \&c.$$

$$\overline{\pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1}} \times \overline{n-r+1} + 1 \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}} \mp$$

$$\overline{\frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1}} \times \overline{n-r} \times \int \frac{x^{n-r-1}\dot{x}}{\sqrt{1-x^2}}.$$

Hence, if the two laſt fluents be given, we have the general law of continuation up to the fluent of $\frac{x^n \dot{x}}{\sqrt{1-x^2}}$, where the index of x without the vinculum increaſes by unity each time. And in the ſame manner we may (by increaſing the index of x without by m) find the fluent of $\frac{x^n \dot{x}}{\sqrt{a^{2m} - x^{2m}}}$ if we have given the fluents of $\frac{x^{n-rm}\dot{x}}{\sqrt{a^{2m} - x^{2m}}}$ and $\frac{x^{n-\overline{r+1}\cdot m}\dot{x}}{\sqrt{a^{2m} - x^{2m}}}$. Thus we have a general law of continuation, where the index of x without is increaſed by half the index under the vinculum.

V. Aſſume $\dot{F} = \frac{x^n \dot{x}}{x^m - b} = x^{n-m}\dot{x} + bx^{n-2m}\dot{x} + b^2 x^{n-3m}\dot{x} + \&c. + \frac{b^r x^{n-rm}\dot{x}}{x^m - b}$, then $F = \frac{x^{n-m+1}}{n-m+1} + \frac{bx^{n-2m+1}}{n-2m+1} + \frac{b^2 x^{n-3m+1}}{n+3m+1} + \&c. + W$, where W is put for the fluent of the laſt term. Now

$$\int x^n \dot{x} \sqrt{\frac{x^m - a}{x^m - b}} = \int \frac{x^n \dot{x}}{x^m - b} \times \sqrt{\overline{x^m - a} \times \overline{x^m - b}} = F \times \sqrt{\overline{x^m - a} \times \overline{x^m + b}}$$

$-\int F \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}$ = (by fubftituting for F its value in the latter quantity, and putting A, B, C, &c. for the co-efficients which arife therefrom) $F \times \sqrt{\overline{x^m - a \times x^m - b}} - A\int \frac{x^n\dot{x}}{\sqrt{x^m - a \times x^m - b}} - B\int \frac{x^{n-m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} - C\int \frac{x^{n-2m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} -$ &c. $-\int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}$. Now $\frac{x^n\dot{x}}{\sqrt{x^m - a \times x^m - b}} = \frac{x^n\dot{x}}{x^m - a}\sqrt{\frac{x^m - a}{x^m - b}}$, $\frac{x^{n-m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} = \frac{x^{n-m}\dot{x}}{x^m - a}\sqrt{\frac{x^m - a}{x^m - b}}$, $\frac{x^{n-2m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} = \frac{x^{n-2m}\dot{x}}{x^m - a}\sqrt{\frac{x^m - a}{x^m - b}}$, &c. But

$$\frac{x^n\dot{x}}{x^m - a} = x^{n-m}\dot{x} + ax^{n-2m}\dot{x} + a^2x^{n-3m}\dot{x} + \text{\&c.} + a^{r-1}x^{n-rm}\dot{x} + \frac{a^r x^{n-rm}\dot{x}}{x^m - a}$$

$$\frac{x^{n-m}\dot{x}}{x^m - a} = x^{n-2m}\dot{x} + ax^{n-3m}\dot{x} + \text{\&c.} + a^{r-2}x^{n-rm}\dot{x} + \frac{a^{r-1}x^{n-rm}\dot{x}}{x^m - a}$$

$$\frac{x^{n-2m}\dot{x}}{x^m - a} = x^{n-3m}\dot{x} + \text{\&c.} + a^{r-3}x^{n-rm}\dot{x} + \frac{a^{r-2}x^{n-rm}\dot{x}}{x^m - a}$$

&c. &c. &c.

Hence, by fubftitution, $\int x^n\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} = F \times \sqrt{\overline{x^m - a} \times \overline{x^m - b}}$ $- A\int x^{n-m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa + B} \times \int x^{n-2m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa^2 + Ba + C}$ $\times \int x^{n-3m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}}$ - &c. $- \overline{Aa^{r-1} + Ba^{r-2} + Ca^{r-3} + \text{\&c.}} \times$ $\times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa^r + Ba^{r-1} + Ca^{r-2} + \text{\&c.}} \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{x^m - a + x^m - b}}$

$-\int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}$. But

$- \int W \times \frac{2mx^{2m-1}\dot{x} - \overline{\quad + b} . mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}$ is $- W \times \sqrt{x^m - a \times x^m - b}$ $+ b^r \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}}$; hence, by substituting this for the last term, it is manifest, that $- W \times \sqrt{x^m - a \times x^m - b}$ will be destroyed by the last term of $F \times \sqrt{x^m - a \times x^m - b}$ when we substitute for F its value; therefore, if we put $M = \frac{x^{n-m+1}}{n-m+1} + \frac{bx^{n-2m+1}}{n-2m+1} + \&c. + \frac{b^{r-1}x^{n-rm+1}}{n-rm+1}$, we have $\int x^n\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} = M \times \sqrt{x^m - a \times x^m - b} - A\int x^{n-m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa + B} \times \int x^{n-2m}\dot{x} \sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa^2 + Ba + C} \times \int x^{n-3m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \&c. - \overline{Aa^{r-1} + Ba^{r-2} + \&c. - b^r} \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa^r + Ba^{r-1} + Ca^{r-2} + \&c.} \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{x^m - a \times x^m - b}}$. Hence if the last two fluents be given, we have the general law of continuation up to the fluent of $x^n\dot{x}\sqrt{\frac{x^m - a}{x^m - b}}$.

The utility of finding fluents by continuation was manifest to Sir ISAAC NEWTON, who first proposed it; and since his time some of the most eminent mathematicians have employed much of their attention upon it. The method which I have investigated and exemplified in this Paper I offer as being entirely new; and at the same time it not only exhibits, at once, the general law up to the required fluent, but also appears, from some of the instances here given, to be more extensive and convenient in its application than any method hitherto offered.

The general reſolution of the given fluxion into a ſeries of fluxions of the ſame kind, where the index of the unknown quantity without the vinculum keeps decreaſing or increaſing either by the index under or by half the index, has not, that I know of, before been given; which furniſhes us at once not only with a very eaſy method of *continuing* fluents, but alſo points out a very ſimple method of inveſtigating the fluent of the given fluxion *without* continuation. For if $\int \dot{A} = p + b \int \dot{B} + c \int \dot{C} + d \int \dot{D} + \&c.$ $\int \dot{B} = p' + c' \int \dot{C} + d' \int \dot{D} + \&c.$ $\int \dot{C} = p'' + d'' \int \dot{D} + \&c.$ &c. &c. then if for $\int \dot{B}$, $\int \dot{C}$, &c. &c. we ſubſtitute their reſpective values, we ſhall get a general ſeries for $\int \dot{A}$ without continuation. The extent of any new method is, at firſt, ſeldom obvious; and how far that which is here propoſed may be ſucceſsfully employed in other caſes will beſt appear from its application. Different methods will always be found to have their uſes in particular caſes; for where one becomes impracticable another will often be found to ſucceed; and I hope that which is here offered will contribute ſomething towards facilitating the inveſtigation of fluents.